Section 2.2 The Inverse of a Matrix

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an inverse of A.

In fact, C is uniquely determined by A. This unique inverse is denoted by A^{-1}, so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is not invertible is sometimes called a singular matrix, and an invertible matrix is called a nonsingular matrix.

Theorem 4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The quantity $ad - bc$ is called the determinant of A, and we write

$$\det A = ad - bc$$

Theorem 4 says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Example 1. Find ~~Let~~ the invese of A if $A = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$. Use the result to solve the equation $A\mathbf{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$.

Ans: $A = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Using Thm 4, we know

$$A^{-1} = \cancel{\frac{1}{3\times 2 - 1\times 7}}\,{-1} \begin{bmatrix} 2 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \xrightarrow{\text{Multiply both sides by } A^{-1}} A^{-1}A\vec{x} = A^{-1}\begin{bmatrix} 5 \\ 11 \end{bmatrix} \xrightarrow{\text{since } A^{-1}A = I} I\vec{x} = A^{-1}\begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Thus $\vec{x} = I\vec{x} = A^{-1}\begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $x_1 = 1$ and $x_2 = 2$

In method in Example 1 works in general cases if A is invertible. Moreover, the solution is unique:

Theorem 5. If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Theorem 6.

a. If A is an invertible matrix, then A^{-1} is invertible and

$$\left(A^{-1}\right)^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Remark: The following generalization of Theorem 6 (b) is needed later.
The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order. E.g. $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

Example 2. Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

ANS: $A = PBP^{-1}$

$$\xrightarrow[\text{both sides by } P^{-1}]{\text{Left-multiply}} \quad \underline{P^{-1}A} = \overbrace{P^{-1}P}^{I \text{ (identity)}}BP^{-1} = IBP^{-1} = \underline{BP^{-1}}$$

$$\Longrightarrow \quad P^{-1}A = BP^{-1}$$

$$\xrightarrow[\text{both sides by } P]{\text{Right-multiply}} \quad P^{-1}AP = B\underbrace{P^{-1}P}_{I \text{ (identity)}} = BI = B$$

Thus $B = P^{-1}AP$

Elementary Matrices

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Example 3. Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A.

ANS: Observe that $I \xrightarrow{(-4)\times R1 + R3} E_1$

$$I \xrightarrow{R_1 \leftrightarrow R_2} E_2$$

$$I \xrightarrow{R_3 \to 5R_3} E_3$$

So E_1, E_2, E_3 are elementary matrices.

We compute

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix},$$

Addition of $(-4)\times R1$ of A to $R3$ produces E_1A

$$E_2 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

Interchanging R1 and R2 produces E_2A

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

$5\times R3$ produces E_3A

Theorem 7. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

proofs on Pg 116. (elementary matrice were used in the proof)

An Algorithm for Finding A^{-1}

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

Example 4. Find the inverses of the given matrix, if it exists. Use the algorithm above.

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}$$

Check Pg 117 for an example about 3x3 matrix.

ANS: $[A \quad I] = \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 7 & 2 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R1 \times (-\frac{7}{3}) + R2} \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{7}{3} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \times (-3)} \left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 1 & 7 & -3 \end{array}\right]$$

$$\xrightarrow{(-1) \times R_2 + R_1} \left[\begin{array}{cc|cc} 3 & 0 & -6 & 3 \\ 0 & 1 & 7 & -3 \end{array}\right]$$

$$\xrightarrow{R_1 \times (\frac{1}{3})} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 7 & -3 \end{array}\right]$$

$$= [I \quad A^{-1}]$$

Thus $A^{-1} = \begin{bmatrix} -2 & 1 \\ 7 & -3 \end{bmatrix}$